

Information metric from Riemannian superspaces.

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Abstract

The Fisher's information metric is introduced in order to find the real meaning of the probability distribution in classical and quantum systems described by Riemannian non-degenerated superspaces. In particular, the physical rôle played by the coefficients \mathbf{a} and \mathbf{a}^* of the pure fermionic part of a genuine emergent metric solution, obtained in previous work [1] is explored. To this end, two characteristic viable distribution functions are used as input in the Fisher definition: first, a Lagrangian generalization of the Hitchin Yang-Mills prescription and, second, the probability current associated to the emergent non-degenerate superspace geometry. Explicitly, we have found that the metric solution of the superspace allows establish a connexion between the Fisher metric and its quantum counterpart, corroborating early conjectures by Caianiello *et al.* This quantum mechanical extension of the Fisher metric is described by the CP^1 structure of the FubiniStudy metric, with coordinates \mathbf{a} and \mathbf{a}^* .

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I. INTRODUCTION

The problem of giving an unambiguous quantum mechanical description of a particle in a general spacetime has been repeatedly investigated. The introduction of supersymmetry provided a new approach to this question, however, some important aspects concerning the physical observables remain not completely understood, classically and quantically speaking.

The superspace concept, on the other hand, simplify considerably the link between ordinary relativistic systems and ‘supersystems’, extending the standard (bosonic) spacetime by means of a general (super)group manifold, equipped with also fermionic (odd) coordinates.

In a previous work [1] we introduced, besides other supersymmetric quantum systems of physical interest, a particular $N = 1$ superspace [2]. That was made with the aim of studying a superworld-line quantum particle (analogously to the relativistic case) and its relation with SUGRA theories [2, 3, 5]. The main feature of this superspace is that the supermetric, which is the basic ingredient of a Volkov-Pashnev particle action [4, 5], is *invertible and non-degenerate*, that is, of $G4$ type in the Casalbuoni's classification [6]. As shown in [2, 3], the non-degeneracy of the supermetrics (and therefore of the corresponding superspaces) leads to important consequences in the description of physical systems. In particular, notorious geometrical and topological effects on the quantum states, namely, *consistent mechanisms of localization and confinement*, due purely to the geometrical character of the Lagrangian. Also *an alternative to the Randall-Sundrum (RS) model without extra bosonic coordinates*, can be consistently formulated in terms of such non degenerated superspace approach, eliminating the problems that the RS-like models present at the quantum level [1, 3].

Given the importance of the non degeneracy of the supermetrics in the formulation of physical theories, in the present article we analyze further the super-line element introduced in [2, 4], focusing on a probabilistic context. To this specific end, the Fisher metric [7] (Fisher-Rao in the quantum sense [8]) have been considered in several works in order to provide a geometrical interpretation of the statistical measures. Fisher's information measure (FIM) was advanced already in the 1920's decade, well before the advent of Information Theory (IT). Much interesting work has been devoted to the physical applications of FIM in recent times (see, for instance, [9] and references therein). In [12], a generalization of the Yang-Mills Hitchin proposal was made suggesting an identification of the Lagrangian density with the Fisher probability distribution ($P(\theta)$). However, this idea was explored from a variational point of view, in previous work by Plastino *et al.* [10, 11] This proposal brought a contribution to the line of works looking for a connexion between the spacetime geometry and quantum field theories.

In the last decades it has been claimed that the above expectation is partially realized in the AdS/CFT (anti-de Sitter/Conformal Field Theory) correspondence [13], which asserts that the equivalence of a gravitational theory (i.e., the geometry of spacetime) and a conformal quantum field theory at the boundary of spacetime certainly exists.

The organization of this paper is as follows: in Section II, the Fisher metric and a new generalization of the Hitchin proposal are introduced. Section III presents global aspects of

the $N = 1$ non-degenerate superspace solution of reference [1]. In Section V we analyze the ‘bosonic’ (B_0) part of our Fisher supermetric by ‘turning off’ all the fermions. Section VI presents further discussions on our results, in connexion with a quantum extension of the Fisher metric. Finally, Section VII is devoted to collect the main results and our concluding remarks.

II. FISHER’S METRIC AND HITCHIN’S PRESCRIPTION

In general, the Fisher information metric [17] (more precisely, the Fisher-Rao metric) is a Riemannian metric for the manifold of the parameters of probability distributions. The Rao distance (geodesic distance in the parameter manifold) provides a measure of the difference between distinct distributions. In the thermodynamic context, the Fisher information metric is directly related to the rate of change in the corresponding order parameters and can be used as an information-geometric complexity measure for classifying phase transitions, *e.g.*, the scalar curvature of the thermodynamic metric tensor diverges at (and only at) a phase transition point (this issue will be analyzed in future work). In particular, such relations identify second-order phase transitions via divergences of individual matrix elements.

The Fisher-Rao information metric is given by [7, 8]

$$G_{ab}(\theta) = \int d^D x P(x; \theta) \partial_a \ln P(x; \theta) \partial_b \ln P(x; \theta) \quad (1)$$

where x^μ ($\mu, \nu = 0, \dots, D$) are the random variables and θ_a ($a, b = 1, \dots, N$) are the parameters of the probability distribution. Besides this, $P(x; \theta)$ must fulfil the normalization condition

$$\int d^D x P(x; \theta) = 1 \quad (2)$$

In his work [14], Hitchin proposed the use of the squared field strength of Yang-Mills theory as a probability distribution. A generalization of the Hitchin’s proposal [12] – see also [10] – consist in identifying the probability distribution with the *on-shell* Lagrangian density of a field theory

$$P(x; \theta) := -\mathcal{L}(x; \theta)|_{\text{solution}}. \quad (3)$$

In Ref. [12], they consider the case in which the distribution dependence on the variables takes the form $P(x, \theta) = P(x - \theta)$. Note that this is only possible in the cases in which the

numbers of the space-time dimensions x and the parameters θ coincide, that is, in which $D = N$.

In the present work we will consider two different approaches to the problem. First, we will follow the ‘generalized Hitchin prescription’, identifying our Lagrangian (calculated at the solution) with the probability distribution. Then, we will introduce a new proposal: we will take the state probability current of the emergent metric solution of the superspace (obtained explicitly in [1]) as being itself the probability distribution. The results of the two approaches will be compared in order to infer the physical meaning of the \mathbf{a} and \mathbf{a}^* parameters appearing in the pure fermionic part of the superspace metric.

III. EMERGENT METRIC SOLUTION

The model introduced in [1] describes a free particle in a superspace with coordinates $z_A \equiv (x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. The corresponding Lagrangian density is

$$\mathcal{L} = -m\sqrt{\omega^A\omega_A} = -m\sqrt{\dot{\omega}_\mu^\circ\dot{\omega}^\circ{}^\mu + \mathbf{a}\dot{\theta}^\alpha\dot{\theta}_\alpha - \mathbf{a}^*\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}}. \quad (4)$$

where $\dot{\omega}_\mu^\circ = \dot{x}_\mu - i(\dot{\theta}^\alpha\sigma_\mu\bar{\theta} - \theta^\alpha\sigma_\mu\dot{\bar{\theta}})$, and the dot indicates derivative with respect to the evolution parameter τ , as usual. In coordinates, the line element of the superspace reads,

$$ds^2 = \dot{z}^A\dot{z}_A = \dot{x}^\mu\dot{x}_\mu - 2i\dot{x}^\mu(\dot{\theta}^\alpha\sigma_\mu\bar{\theta} - \theta^\alpha\sigma_\mu\dot{\bar{\theta}}) + (\mathbf{a} - \bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}})\dot{\theta}^\alpha\dot{\theta}_\alpha - (\mathbf{a}^* + \theta^\alpha\theta_\alpha)\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}} \quad (5)$$

The ‘squared’ solution with three compactified dimensions (λ spin fixed) is [1]

$$g_{AB}(t) = e^{A(t)+\xi\varrho(t)}g_{AB}(0), \quad (6)$$

where the initial values of the metric components are given by

$$g_{ab}(0) = \langle\psi(0)| \begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab} |\psi(0)\rangle, \quad (7)$$

or, explicitly,

$$\begin{aligned} g_{\mu\nu}(0) &= \eta_{\mu\nu}, & g_{\mu\alpha}(0) &= -i\sigma_{\mu\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}, & g_{\mu\dot{\alpha}}(0) &= -i\theta^\alpha\sigma_{\mu\alpha\dot{\alpha}}, \\ g_{\alpha\beta}(0) &= (a - \bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}})\epsilon_{\alpha\beta}, & g_{\dot{\alpha}\dot{\beta}}(0) &= -(a^* + \theta^\alpha\theta_\alpha)\epsilon_{\dot{\alpha}\dot{\beta}}. \end{aligned} \quad (8)$$

It is worth mention here that these components were obtained in a simpler case in [5].

The bosonic and spinorial parts of the exponent in the superfield solution (6) are, respectively,

$$A(t) = - \left(\frac{m}{|\mathbf{a}|} \right)^2 t^2 + c_1 t + c_2, \quad (9)$$

and

$$\begin{aligned} \xi \varrho(t) &= \xi (\phi_\alpha(t) + \bar{\chi}_{\dot{\alpha}}(t)) \\ &= \theta^\alpha \left(\overset{\circ}{\phi}_\alpha \cos(\omega t/2) + \frac{2}{\omega} Z_\alpha \right) - \bar{\theta}^{\dot{\alpha}} \left(-\overset{\circ}{\bar{\phi}}_{\dot{\alpha}} \sin(\omega t/2) - \frac{2}{\omega} \bar{Z}_{\dot{\alpha}} \right) \\ &= \theta^\alpha \overset{\circ}{\phi}_\alpha \cos(\omega t/2) + \bar{\theta}^{\dot{\alpha}} \overset{\circ}{\bar{\phi}}_{\dot{\alpha}} \sin(\omega t/2) + 4|\mathbf{a}| \text{Re}(\theta Z), \end{aligned} \quad (10)$$

where $\overset{\circ}{\phi}_\alpha, Z_\alpha, \bar{Z}_{\dot{\beta}}$ are constant spinors, $\omega \approx 1/|\mathbf{a}|$ and the constant c_1 , due to the obvious physical reasons and the chirality restoration of the superfield solution [1, 2], should be taken purely imaginary.

IV. FISHER INFORMATION METRIC FROM RIEMANNIAN SUPERSPACES

Fisher method considers a family of probability distributions, characterized by certain number of parameters. The metric components are then defined by considering derivatives in different ‘directions’ in the parameters space. That is, measuring ‘how distant’ two distinct set of parameters put apart the corresponding probability distributions.

In the following we will calculate the Fisher information metric corresponding to a generalized Hitchin ‘on-shell’ Lagrangian prescription. In our case, the parameters of interest in the metric solution (6) are \mathbf{a} and \mathbf{a}^* , which could indicate the residual effects of supersymmetry given that they survive even when ‘turning off’ all the fermionic fields.

A. Generalized Hitchin prescription for the probability distribution

Following the generalized Hitchin prescription we identify the probability distribution with Lagrangian (4), evaluated at solution (6), $g_{AB}(t) = e^{(A(t) + \xi \varrho(t))} g_{AB}(0)$. Thus, the probability distribution density takes the form

$$P(z^A, a, a^*) := -\mathcal{L}|_{g_{AB}(t)} = e^{\frac{1}{2}(A(t) + \xi \varrho(t))} \mathcal{L}_0, \quad (11)$$

where

$$\mathcal{L}_0 \equiv \mathcal{L}(g_{ab}(0)) = m \sqrt{\overset{\circ}{\omega}^\nu \overset{\circ}{\omega}^\mu + \mathbf{a} \dot{\theta}^\alpha \dot{\theta}_\alpha - \mathbf{a}^* \dot{\bar{\theta}}^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}}}. \quad (12)$$

Let us now calculate the \mathbf{a} and \mathbf{a}^* derivatives of our probability distribution

$$\frac{\partial P}{\partial \mathbf{a}} = \mathcal{L}_0 \left[\frac{1}{2} \frac{\partial(A(t) + \xi \varrho(t))}{\partial \mathbf{a}} + \frac{1}{\mathcal{L}_0} \frac{\partial \mathcal{L}_0}{\partial \mathbf{a}} \right] e^{\frac{1}{2}(A(t) + \xi \varrho(t))}, \quad (13)$$

$$\frac{\partial P}{\partial \mathbf{a}^*} = \mathcal{L}_0 \left[\frac{1}{2} \frac{\partial(A(t) + \xi \varrho(t))}{\partial \mathbf{a}^*} + \frac{1}{\mathcal{L}_0} \frac{\partial \mathcal{L}_0}{\partial \mathbf{a}^*} \right] e^{\frac{1}{2}(A(t) + \xi \varrho(t))}. \quad (14)$$

The last terms in the squared brackets give

$$\frac{1}{\mathcal{L}_0} \frac{\partial \mathcal{L}_0}{\partial \mathbf{a}} = \frac{m^2}{2\mathcal{L}_0^2} \dot{\theta}^\alpha \dot{\theta}_\alpha \quad \text{and} \quad \frac{1}{\mathcal{L}_0} \frac{\partial \mathcal{L}_0}{\partial \mathbf{a}^*} = -\frac{m^2}{2\mathcal{L}_0^2} \dot{\bar{\theta}}^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}} \quad (15)$$

From (10) we have

$$\begin{aligned} \Xi(t; |a|) &\equiv \frac{\partial(A(t) + \xi \varrho(t))}{\partial |\mathbf{a}|} \\ &= \frac{2m^2}{|\mathbf{a}|^3} t^2 + \frac{\omega^2 t}{2} \left(\theta^\alpha \overset{\circ}{\phi}_\alpha \sin(\omega t/2) - \bar{\theta}^{\dot{\alpha}} \overset{\circ}{\bar{\phi}}_{\dot{\alpha}} \cos(\omega t/2) \right) + 4\text{Re}(\theta Z), \end{aligned} \quad (16)$$

Finally, putting $|\mathbf{a}| = \sqrt{\mathbf{a}\mathbf{a}^*}$ we have that

$$\frac{\partial |\mathbf{a}|}{\partial \mathbf{a}} = \frac{\mathbf{a}^*}{2|\mathbf{a}|} \quad \text{and} \quad \frac{\partial |\mathbf{a}|}{\partial \mathbf{a}^*} = \frac{\mathbf{a}}{2|\mathbf{a}|}, \quad (17)$$

We can now write down the Fisher's metric components

$$\begin{aligned} G_{aa} &= \int dx^4 P^{-1} \left[\frac{\partial P}{\partial \mathbf{a}} \right]^2 = \frac{\mathcal{L}_0}{16} \int dt \left(\frac{\mathbf{a}^*}{|\mathbf{a}|} \Xi(t; |a|) + \frac{2m^2}{\mathcal{L}_0^2} \dot{\theta}^\alpha \dot{\theta}_\alpha \right)^2 e^{\frac{1}{2}(A(t) + \xi \varrho(t))} \\ G_{a^*a^*} &= \int dx^4 P^{-1} \left[\frac{\partial P}{\partial \mathbf{a}^*} \right]^2 = \frac{\mathcal{L}_0}{16} \int dt \left(\frac{\mathbf{a}}{|\mathbf{a}|} \Xi(t; |a|) - \frac{2m^2}{\mathcal{L}_0^2} \dot{\bar{\theta}}^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}} \right)^2 e^{\frac{1}{2}(A(t) + \xi \varrho(t))} \\ G_{aa^*} &= g_{a^*a} = \int dx^4 P^{-1} \frac{\partial P}{\partial \mathbf{a}} \frac{\partial P}{\partial \mathbf{a}^*} = \frac{\mathcal{L}_0}{16} \int dt \left(\Xi(t; |a|)^2 + 4 \frac{m^4}{\mathcal{L}_0^4} \dot{\theta}^\alpha \dot{\theta}_\alpha \dot{\bar{\theta}}^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}} \right) e^{\frac{1}{2}(A(t) + \xi \varrho(t))} \end{aligned} \quad (18)$$

B. State probability current as distribution

Our second approach consist in identifying the state probability density (zero component of the probability current) of the solution (6) as the probability density itself.

The zero component of the probability current can be obtained by making

$$j_0(t) = 2E^2 g_{ab}(t) g^{ab}(t). \quad (19)$$

Then, putting $\mathcal{K}_0 \equiv 32E^2 |\alpha|^2$, we have

$$j_0(t) = \frac{1}{16} \mathcal{K}_0 e^{-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi \varrho(t)}, \quad (20)$$

and

$$\frac{\partial j_0}{\partial |\mathbf{a}|} = \frac{1}{8} \mathcal{K}_0 \Xi(t; |a|) e^{-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi \varrho(t)}. \quad (21)$$

Therefore, taking $P \equiv j_0(t)$ and using again (17) we get

$$G_{aa} = \int dx^4 P^{-1} \left[\frac{\partial P}{\partial \mathbf{a}} \right]^2 = \left(\frac{\mathbf{a}^*}{|\mathbf{a}|} \right)^2 \mathcal{I}_\Xi \quad (22)$$

$$G_{a^*a^*} = \int dx^4 P^{-1} \left[\frac{\partial P}{\partial \mathbf{a}^*} \right]^2 = \left(\frac{\mathbf{a}}{|\mathbf{a}|} \right)^2 \mathcal{I}_\Xi \quad (23)$$

$$G_{aa^*} = G_{a^*a} = \int dx^4 P^{-1} \frac{\partial P}{\partial \mathbf{a}} \frac{\partial P}{\partial \mathbf{a}^*} = \mathcal{I}_\Xi, \quad (24)$$

where \mathcal{I}_Ξ corresponds to the integral of the temporal part

$$\mathcal{I}_\Xi \equiv \frac{1}{16} \mathcal{K}_0 \int dt \Xi(t; |a|)^2 e^{-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi \varrho(t)}. \quad (25)$$

V. THE B_0 PARTE OF THE FISHER SUPERMETRIC

A. Generalized Hitchin prescription with zero fermions

When putting all fermion to zero, the derivative of the time dependent exponential and the \mathcal{L}_0 initial value ‘on-shell’ Lagrangian reduce, respectively, to

$$\Xi(t; |a|)|_{\theta=\chi=0} = \frac{2m^2}{|\mathbf{a}|^3} t^2 \quad \text{and} \quad \mathcal{L}_0|_{\theta=\chi=0} = m \sqrt{\dot{x}^\mu \dot{x}_\mu} = m, \quad (26)$$

where the last equality makes explicit the relativistic constraint $v^2 \equiv \dot{x}^\mu \dot{x}_\mu = 1$.

In that case, and writing the complex parameters as $\mathbf{a} = |\mathbf{a}|e^{i\phi}$, the metric components take the simple form

$$G_{ab} = \mathcal{I}(m, c_1, c_2, |\mathbf{a}|) \begin{pmatrix} e^{-i2\phi} & 1 \\ 1 & e^{i2\phi} \end{pmatrix}, \quad (27)$$

where indices a, b take values in $\{\mathbf{a}, \mathbf{a}^*\}$, and the prefactor is the integral of the time varying factor, that can be easily performed to obtain

$$\begin{aligned} \mathcal{I}(m, c_1, c_2, |\mathbf{a}|) &\equiv \frac{1}{4} \left(\frac{m^5}{|\mathbf{a}|^6} \right) \int dt t^4 e^{-\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1 t + c_2} \\ &= \frac{\sqrt{\pi}}{64} \left[\left(\frac{c_1}{m} \right)^4 |\mathbf{a}|^3 + 12 \left(\frac{c_1}{m} \right)^2 |\mathbf{a}| + 12 |\mathbf{a}|^{-1} \right] e^{\frac{1}{4} \left(\frac{c_1}{m} \right)^2 |\mathbf{a}|^2 + c_2} \end{aligned} \quad (28)$$

Note that the metric (27) above can also be put in terms of Pauli’s matrices as

$$G_{ab} = \mathcal{I}(m, c_1, c_2, |\mathbf{a}|) (\mathbb{I}_2 \cos(\phi) - i \sin(\phi) \sigma_z + \sigma_x) \quad (29)$$

B. State probability current with zero fermions

Turning off all the fermions in solution (20), the zero component of the probability distribution current reduces to

$$j_0|_{\theta=\chi=0} = \frac{1}{16} \mathcal{K}_0 e^{-2\frac{m^2}{|\mathbf{a}|^2}t^2+2c_2}, \quad (30)$$

and its derivative to

$$\left. \frac{\partial j_0}{\partial |\mathbf{a}|} \right|_{\theta=\chi=0} = \frac{1}{4} \frac{m^2}{|\mathbf{a}|^3} \mathcal{K}_0 t^2 e^{-2\frac{m^2}{|\mathbf{a}|^2}t^2+2c_2}. \quad (31)$$

Therefore, the metric components in this case take the simple form

$$G_{ab} = \mathcal{J}(m, E, |\alpha|, c_2, |\mathbf{a}|) \begin{pmatrix} e^{-i2\phi} & 1 \\ 1 & e^{i2\phi} \end{pmatrix}. \quad (32)$$

Again, a, b taking values in $\{\mathbf{a}, \mathbf{a}^*\}$, and now the prefactor is obtained performing the integral

$$\begin{aligned} \mathcal{J}(m, E, |\alpha|, c_2, |\mathbf{a}|) &\equiv \frac{1}{4} \left(\frac{\mathcal{K}_0 m^4}{|\mathbf{a}|^6} \right) \int dt t^4 e^{-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2} \\ &= \frac{3\sqrt{2\pi}}{4} \frac{E^2 |\alpha|^2}{m} e^{2c_2} |\mathbf{a}|^{-1}, \end{aligned} \quad (33)$$

where in last equality we have made explicit the value of \mathcal{K}_0 .

Note that this last expression, in contrast with eq. (28), presents only the $|\mathbf{a}|^{-1}$ singular term. This is precisely due to the lack of the ‘free wave’ (linear in t) term in the exponential factor, which leads to a complete departure of the Gaussian behaviour shown by (28), remaining in common just the singular term.

VI. INFORMATION METRIC AND GEOMETRICAL LAGRANGIANS: FURTHER DISCUSSION

Note that the Fisher metric can be rewritten in the form

$$\begin{aligned} G_{ab}(\theta) &= \int d^D x P(x; \theta) \partial_a \ln P(x; \theta) \partial_b \ln P(x; \theta) \\ &= 4 \int d^D x \partial_a P^{1/2}(x; \theta) \partial_b P^{1/2}(x; \theta). \end{aligned} \quad (34)$$

The appearance of the square root of the probability density P above, naturally leads to the identification

$$P^{1/2} \equiv \mathcal{L}_g \quad \Rightarrow \quad P = ds^2 \quad (35)$$

Under this point of view, the P function is related to the line element that define the geometrical (superspace in our case) Lagrangian of the theory. Therefore, this is a first approach to connect the two “distances”: the Rao distance in the probability parameters manifold, and the geometric space-time distance.

A. P as the current of probability: the quantum correspondence

Consider a Hilbert space \mathcal{H} with a symmetric inner product G_{ij} . For instance, we can have in mind the case $\mathcal{H} = \mathbf{L}_2(\mathcal{R})$, where $\mathcal{R} = \mathbb{R}^{2m}$ (e.g.: the phase space of a classical dynamical system, the configuration space spin systems, etc.) The equation (34), as was first observed by Caianiello et al. [15], puts in evidence the clear possibility of mapping the probability density function $P(x; \theta)$ on \mathcal{R} to \mathcal{H} by forming the square-root.

As we propose in the present paper, using directly (and precisely) the probability current j_0 , of the metric state solution $g_{AB}(t)$, as the probability density $P(x; \theta)$,

$$j_0 = \frac{1}{16} \mathcal{K}_0 e^{-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi \varrho(t)} \equiv P(x; \theta), \quad (36)$$

leads to an identification (intuited by Caianiello in [15]) that can be immediately implemented and effectively realized. The metric components take then the form

$$\begin{aligned} G_{ab}(\theta) &= 4 \int d^D x \partial_a P^{1/2}(x; \theta) \partial_b P^{1/2}(x; \theta) \\ &\rightarrow 4 \int d^D x \partial_a g_{AB}(x; \mathbf{a}, \mathbf{a}^*) \partial_b g^{AB}(x; \mathbf{a}, \mathbf{a}^*) \\ &= 4 \int d^D x \partial_a g_{AB}(x; \mathbf{a}, \mathbf{a}^*) \partial_b g_{CD}(x; \mathbf{a}, \mathbf{a}^*) \eta^{(AB)(CD)}. \end{aligned} \quad (37)$$

Then, the quantum ‘crossover’ is

$$\begin{aligned} G_{ab}(\theta) &= 4 \partial_a g_{AB}(x; \mathbf{a}, \mathbf{a}^*) \partial_b g_{CD}(x; \mathbf{a}, \mathbf{a}^*) \eta^{(AB)(CD)} \\ &\equiv \langle \partial_a g_{AB}(x; \mathbf{a}, \mathbf{a}^*) \partial_b g_{CD}(x; \mathbf{a}, \mathbf{a}^*) \rangle. \end{aligned} \quad (38)$$

The resemblance with a Sigma model is quite evident. Even more, the ‘natural’ geometrical normalization condition $g_{AB}(x; \mathbf{a}, \mathbf{a}^*) g^{AB}(x; \mathbf{a}, \mathbf{a}^*) = D$ (regarding the genuine emergent origin of the spin 2 state $g_{AB}(x; \mathbf{a}, \mathbf{a}^*)$), suggests a pseudospherical constraint in \mathcal{H} . However, the above relation (38) is not the more general possibility of quantum extension

of the Fisher's metric. We can also introduce the following Hermitian metric tensor

$$\begin{aligned}\tilde{G}_{ab}(\theta) = & \langle \partial_a g_{AB}(x; \mathbf{a}, \mathbf{a}^*) \partial_b g_{CD}(x; \mathbf{a}, \mathbf{a}^*) \rangle \\ & - \langle \partial_a g_{AB}(x; \mathbf{a}, \mathbf{a}^*) g_{CD}(x; \mathbf{a}, \mathbf{a}^*) \rangle \langle g_{AB}(x; \mathbf{a}, \mathbf{a}^*) \partial_b g_{CD}(x; \mathbf{a}, \mathbf{a}^*) \rangle,\end{aligned}\quad (39)$$

since its real part can be *exactly* rewritten as [15]

$$\text{Re } \tilde{G}_{ab}(\theta) = \left\langle \partial_a L_{AB}^{1/2}(x; \mathbf{a}, \mathbf{a}^*) \partial_b L_{CD}^{1/2}(x; \mathbf{a}, \mathbf{a}^*) \right\rangle_{HS}, \quad (40)$$

where L_{AB} (non-diagonal representation) [2, 16] is given by

$$\begin{aligned}L_{AB} = & \int \frac{d^2\alpha}{\pi} \left[\int \frac{d^2w}{\pi} \int \frac{d^2\alpha'}{\pi} e^{-\left(\frac{m}{\sqrt{2}|\mathbf{a}|}\right)^2 [(\alpha+\alpha^*)-B]^2+D} e^{\xi\varrho(\alpha'+\alpha'^*)} |f(\xi)|^2 \times \\ & \left(\begin{array}{c} \alpha' \\ \alpha'^* \end{array} \right)_{(m+n)AB} e^{\frac{|w|^2}{4}} e^{\frac{i}{2}[(\alpha-\alpha')w^*+(\alpha^*-\alpha'^*)w]} \right] |\Psi_m(\alpha)\rangle \langle \Psi_n(\alpha')|\end{aligned}\quad (41)$$

with $m, n = 1/4, 3/4$ ($m \neq n$), and \langle, \rangle_{GS} standing for the customary inner product in the Banach space of the Gram-Schmidt operators in \mathcal{H} . The complex numbers α and α^* in the exponential factor are the *eigenvalues* of the coherent states.

From the above expression, the corresponding Gram-Schmidt operator reads

$$G = \int \frac{d^2\alpha}{\pi} \left[\int \frac{d^2w}{\pi} e^{\frac{|w|^2}{4}} e^{\frac{i}{2}[(\alpha-\alpha')w^*+(\alpha^*-\alpha'^*)w]} \right] |\Psi_m(\alpha)\rangle \langle \Psi_n(\alpha')| \quad (42)$$

In summary, our results come to realize the connexion conjectured by Caianiello.

B. Reparameterization invariant formulation (going towards first principles)

In information geometry, reparameterization invariance can be seen as a change of coordinates in a Riemannian manifold, the intrinsic properties of the curvature remaining unchanged in different parameterizations.

The Fisher metric in its original form, is not invariant under reparameterizations. Why it is important to have reparameterization invariance? If we can establish a geometrical relation between the probability distribution, P , and the metric state solution, $g_{ab}(t)$ (or the geometrical Lagrangian) via the probability current, it would be desirable to have reparameterization invariance with respect to the evolution parameter of the physical system, in order to have an unique perspective of evolution of the system. One can go further in such

direction and propose the following (Nambu-Goto inspired) metric

$$G_{ab} = 4 \int d^D x \sqrt{\partial_a P^{1/2}(x; \theta) \partial_b P^{1/2}(x; \theta)}, \quad (43)$$

that is full invariant under reparameterization with respect to the physical evolution of the system. This possibility will be explored in future work.

VII. CONCLUDING REMARKS

In this work we have analyzed several aspects of the geometrical meaning of the Fisher's metric definition. A new generalization of the Hitchin prescription for constructing the Fisher's information metric was presented, taking a geometrical Lagrangian as probability distribution ($P \leftrightarrow \mathcal{L}$). The results were confronted with a completely different prescription: to take as probability distribution the state probability density of an emergent metric (coherent) state, which is a solution for a non-degenerated superspace obtained in a previous work ($P \leftrightarrow j_0$). We then analyze the bosonic (B_0) part of the Fisher supermetric by putting all fermionic fields to zero, in order to compare our solutions with those in the literature.

The main results of this research can be summarized as follows:

i) The choice of the complex constants \mathbf{a} and \mathbf{a}^* as our set of (physically meaningful) parameters is based on that they are responsible for the localized Gaussian behaviour of the physical states. This lead to a Fisher's metric on a complex 2-dimensional manifold presenting notably different behaviours in the two approaches. In the first one ($P \leftrightarrow \mathcal{L}$) the Gaussian behaviour of the metric state solution $g_{ab}(t)$ was preserved while in the second one ($P \leftrightarrow j_0$) it was completely lost. However, it is important to remark that, in both cases, the ultralocal characteristic behavior of $g_{ab}(t)$ is preserved through a singular ($|\mathbf{a}|^{-1}$) term.

ii) In principle, it should be possible to relate, from the quantum point of view, the \mathbf{a} - \mathbf{a}^* complex manifold (Fisher's) metric with an invariant metric on a Kähler or on a projective Hilbert space (CP^1).

iii) As shown in Section VIA, the function P , in sharp contrast with the Hitchin's proposal, can be put in direct relation with the spacetime line element ds^2 by making the identification $P^{1/2} \equiv \mathcal{L}_g$ in the same Fisher's formula.

iv) Also in Sec. VIA we demonstrate that, using the probability current j_0 as the probability density P , the quantum counterpart of the Fisher's metric can be exactly imple-

mented, and all the quantum operators involved in the geometrical correspondence, exactly constructed, as already inferred on a general basis by Caianiello *et al.* in [15].

VIII. ACKNOWLEDGMENTS

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